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On the Tensor Product Theorem for Algebraic Groups*

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I. In this paper we give an alternate development of the well-known tensor product theorem for irreducible representations of semisimple algebraic groups in characteristic p. This theorem was first proved in general by Steinberg [8] using results of Curtis [4]; an exposition appears in Borel's article in [1]. Special cases of the theorem were obtained previously by Brauer and Nesbitt [2], Mark [7], and Wong [10]. Wong later completed his argument in [11] and [12].

In his proof Steinberg first adapted Curtis' results to show that the irreducible representations of the Lie algebra of G lift to projective representations of G, then directly showed by a computation that the appropriate tensor products are irreducible. His point of view was mainly that of Chevalley's Tôhoku paper, in which the algebraic groups were constructed by reduction mod p. Wong's proof also used reduction mod p, deducing the theorem from its validity for Steinberg modules. Both methods yielded information about the irreducible representations of the finite Chevalley groups.

In our version we stick to the algebraic groups and obtain a proof which involves no computations or reduction mod p. The main simplification occurs in part (a) of the proof of Theorem 2; its philosophy has its roots in the theory of group schemes (cf. remarks at end of paper), and yields new results in that theory. Assuming only basic concepts of [6], our proof is self-contained. Because our treatment does contain the necessary adaptation of Curtis' results, this paper can be used to shortcut the treatment of the finite groups in [1, 8, 9].

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II. Let k_0 be the prime subfield of an algebraically closed field k of positive characteristic p. Let G be a simply connected, semisimple algebraic group defined over k_0. The group multiplication on G endows the affine coordinate ring A of G with the structure of a Hopf algebra with comultiplication $\Delta: A \to A \otimes A$. Recall that the Lie algebra $L(G)$ of G is the set of k-derivations $D: A \to A$ satisfying the identity $(1 \otimes D)\Delta = \Delta D$. It has dimension equal to that of G and is *restricted* in the sense that $D^p \in L(G)$ whenever $D \in L(G)$. We let $\mathcal{U}$ denote the restricted enveloping algebra of $L(G)$ (the largest quotient of the usual enveloping algebra in which D^p above is the pth power of D in $\mathcal{U}$). The adjoint action of the group G on $L(G)$ induces an adjoint action of G on $\mathcal{U}$ by Hopf algebra automorphisms. As such $\mathcal{U}$ is a finite-dimensional rational G-module. Any rational G-module V is a restricted $L(G)$-module, and so a $\mathcal{U}$-module; these module structures are compatible with the actions of G on $L(G)$ and $\mathcal{U}$.

The following theorem is closely related to results of Curtis [4].

THEOREM 1. *Any irreducible $\mathcal{U}$-module V extends uniquely to a rational G-module.*

Proof. Let J denote the Jacobson radical of $\mathcal{U}$. Since G is connected, it must fix elementwise the finitely many primitive central idempotents of algebra $\mathcal{U}/J$. Therefore by the structure theory of semisimple algebras, we obtain a homomorphism $\rho: G \to \mathrm{Aut}(\mathrm{End}(V)) = PGL(V)$ of algebraic groups. Since G is simply connected, ρ lifts to a homomorphism $\tilde{\rho}: G \to GL(V)$.[1] It remains only to check that the induced action of $\mathcal{U}$ on V agrees with the original $\mathcal{U}$-action on V. Denote the original action of $\mathcal{U}$ on V by τ, and let τ^* denote the contragredient action. The representation $\tau^* \otimes \tau$ in $V^* \otimes V \cong \mathrm{End}(V)$ agrees with the adjoint action ρ of G; the latter is also given by conjugation in $\mathrm{End}(V)$ by $\tilde{\rho}$. So for $X \in L(G)$, we have $\mathrm{ad}(\tau(X)) = \mathrm{ad}(\tilde{\rho}(X))$. It follows that $\tau(X) = \tilde{\rho}(X) + \mu(X)$ where $\mu: L(G) \to \mathrm{Center}(\mathrm{End}(V))$ is a homomorphism of restricted Lie algebras.

The proof of existence is now completed by the following.

LEMMA. *Let $\mu: L(G) \to k$ be a restricted Lie algebra homomorphism (where the p-operator on k is $x^{[p]} = x^p$). Then $\mu = 0$.*

Proof. Since G is simply connected, its Lie algebra is spanned by the Lie algebras of subgroups isomorphic to SL_2. But it is easy to check that $L(SL_2)$ is generated by elements X with $X^{[p]} = 0$. The result is now clear.

To complete the proof of the theorem, suppose V_1, V_2 rational irreducible

[1] The argument to this point is the same as that in [3]. It seems to be well known that the proof of théorème 2 given there contains a gap; the representations of one-parameter subgroups by truncated exponentials (e.g., formula (9) in [3]) need not hold.

G-modules which are $\mathscr{U}$-isomorphic. Since G acts on $\mathrm{Hom}_{\mathscr{U}}(V_1, V_2) \cong k$ and G has no linear characters, any $\mathscr{U}$-isomorphism between V_1 and V_2 is a G-isomorphism. Q.E.D.

Now let T be a maximal torus of G, and let Δ be a system of simple roots for T relative to a Borel subgroup $B \supseteq T$. A weight μ of T is in the *restricted range* if $0 \leqslant \langle \mu, \alpha^\vee \rangle < p$ for each $\alpha \in \Delta$. For a dominant weight λ, we let $\lambda = \lambda_0 + p\lambda_1 + \cdots + p^n\lambda_n$ be the p-adic expansion of λ, so that each λ_i is in the restricted range. Let $S(\lambda)$ be the irreducible rational G-module of high weight λ. Finally, for a rational G-module W, we let $W^{(p^r)}$ be the module W "twisted" by the rth power of the Frobenius morphism $\sigma: G \to G$. Thus $W^{(p^r)} = W$ as a vector space, but $g \in G$ acts on $W^{(p^r)}$ via g^{σ^r}. Observe that $S(\lambda)^{(p)} \cong S(p\lambda)$ by the highest weight classification, and note also that $L(G)$ acts trivially on $S(\lambda)^{(p)}$ since the differential of the Frobenius morphism is zero.

THEOREM 2 (Steinberg [8]). *For λ dominant, we have*

$$S(\lambda) \cong S(\lambda_0) \otimes S(\lambda_1)^{(p)} \otimes \cdots \otimes S(\lambda_m)^{(p^m)}.$$

Proof. (a) Let S be an irreducible module for G. Then S contains an irreducible $L(G)$-submodule S_1 which, by Theorem 1, has a unique extension (also denoted S_1) to a G-module. Now $\mathrm{Hom}_{L(G)}(S_1, S)$ is a G-module, and the map $\mathrm{Hom}_{L(G)}(S_1, S) \otimes S_1 \to S$, defined by sending $\phi \otimes s \mapsto \phi(s)$ for $\phi \in \mathrm{Hom}_{L(G)}(S_1, S)$ and $s \in S_1$, is a nonzero G-homomorphism. It is, therefore, surjective. But clearly the dimension of $\mathrm{Hom}_{L(G)}(S_1, S) \otimes S_1$ is at most that of S. Hence we have an isomorphism.[2] It follows also that $\mathrm{Hom}_{L(G)}(S_1, S)$ is an irreducible G-module.

(b) Now consider $S(\lambda)$ for λ in the restricted range. By (a) we can write $S(\lambda) = S(\lambda') \otimes S(\mu)$ where $L(G)$ acts trivially on $S(\mu)$ (and in particular $\mu|_{L(T)} = 0$). Comparing high weights, we obtain that $\lambda = \lambda' + \mu$ whence μ lies in the restricted range. Thus, $\mu|_{L(T)} = 0$ implies $\mu = 0$, and we obtain that $S(\lambda)$ is $L(G)$-irreducible.

(c) Finally, for a general dominant weight λ, write $\lambda = \lambda_0 + p\theta$ with λ_0 in the restricted range and θ dominant. Clearly $S(\lambda)$ is a G-composition factor of $S(\lambda_0) \otimes S(\lambda)^{(p)}$. All $L(G)$-composition factors of the latter are $L(G)$-isomorphic to $S(\lambda_0)$ by (b), hence $S(\lambda_0)$ is an irreducible $L(G)$-submodule of $S(\lambda)$. By (a), $S(\lambda) = S(\lambda_0) \otimes S(\psi)$ for some dominant weight ψ. Since $\lambda = \lambda_0 + p\psi$, we

[2] We have learned from J. C. Jantzen that he has independently discovered this same argument in unpublished work. Both Jantzen's interest and ours was stimulated by Ballard's paper, Injective modules for restricted enveloping algebras, *Math. Z.* **163** (1978), 57–63.

Added in proof. There is yet another proof of Theorem 2 in Donkin's Warwick thesis (1977), modulo the Curtis results. Donkin shows $S(\lambda_0) \otimes \cdots$ is irreducible with Hopf algebra methods [13, 2.1].

have $\psi = p\theta$, hence $S(\lambda) = S(\lambda_0) \otimes S(\theta)^{(p)}$. The proof is completed by an obvious induction applied to $S(\theta)$.

From the proof of the theorem and Theorem 1, we obtain the following

COROLLARY (Curtis [4]). *The $S(\lambda)$, λ in the restricted range, are exactly the restricted irreducible $L(G)$-modules. They are distinct for distinct λ.*

Finally, we remark that the approach can be reformulated even more conceptually in the language of affine k-group schemes [5]. For instance, the category of restricted $L(G)$-modules is equivalent to that of rational G_1-modules where G_1 denotes the first infinitesimal subgroup of G. The group scheme G_1 is the kernel of the Frobenius endomorphism in the category of affine k-groups and $G/G_1 \simeq G$. Once one knows that irreducible G_1-modules extend to G, it is entirely obvious from standard group theoretic techniques (Clifford theory) that one should have a tensor product decomposition for an irreducible G-module.

Moreover, as the proof of Theorem 2 shows, there is a tensor product decomposition for the irreducible rational modules of any k-subgroup scheme of G containing G_1. (For example, every irreducible rational BG_1-module has the form $V \otimes p\mu$, where V is an irreducible G-module of restricted high weight and μ is a character on B.) We hope to pursue this later.

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